Simplifications of Triphasic Mixture Theory for Articular Cartilage

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INTRODUCTION

The triphasic mixture theories for articular cartilage [1] have been successfully used to describe the flow-dependent and flow-independent viscoelastic behaviors, swelling behaviors, and electrokinetic behaviors of charged-hydrated soft tissues in the last two decades. It is considered in the biomechanics literature as a unified theory for such materials. However, the mathematical complexity of this constitutive law has limited its applications in cartilage biomechanics studies, especially for those testing setups involving complicated loading conditions and deformation fields (e.g., indentation). The objective of this study is to develop three sets of new triphasic formulations that can greatly simplify the triphasic analysis for: 1) the tissue response at final steady state; 2) the transient response of cartilage under mechanical loading; and 3) the equilibrium response of the tissue under dynamic loadings.

METHODS

<u>The correspondence principle for equilibrium response</u>: By linearizing the triphasic constitutive equations with regular perturbation method [2], it is found that the deformational behavior of a charged-hydrated material at final steady state is identical to that of a pure elastic medium. The mechanical properties of this equivalent material can be correlated with the intrinsic elastic moduli, fixed charge density (FCD), and external solution concentration by following equations:

$$\lambda_s^* = \lambda_s + \Pi \text{, and } \Pi = RT\left(c_0^F\right)^2 / \left(\varphi_0^w \sqrt{\left(c_0^F\right)^2 + 4\left(c^*\right)^2}\right), \quad (1)$$

$$\mu_s^* = \mu_s \,. \quad (2)$$

Here the * quantities denote variables that are the equivalent Lame's coefficients, while λ_s and μ_s are intrinsic Lame's constants defining the porous-permeable elastic solid matrix in the triphasic theory. In the equation for the Donnan osmotic effect Π, R is universal gas constant, T is absolute temperature, φ_0^w is water volume fraction, c_0^F is fixed charge density in the initial free swelling state, and c^* is the external solution concentration. The validity of these simple formulas is independent of the deformation state of the solid matrix. Therefore they can be employed for any loading condition involving arbitrary deformation field (e.g., confined or unconfined compression, tension, permeation, and indentation tests). This makes the employment of triphasic analysis as straightforward as using an elastic model to solve equilibrium problems.

<u>Simplified governing equations for transient response:</u>

$$\nabla^2\phi = 0, \qquad \nabla^2\psi = e$$
$$\frac{\partial e}{\partial t} = A_1\nabla^2 e - A_2\nabla^2\gamma, \quad \frac{\partial \gamma}{\partial t} = A_4\nabla^2\gamma - A_5\nabla^2 e \quad (3\text{-}6)$$

Using the potential functions defined in [3] for the biphasic indentation problem, and the regular perturbation method described above, the governing equations of triphasic mixture theory for cartilage can be linearized as Eqs. (3-6). Here ϕ and ψ are potential functions, e is the dilation of the solid matrix, and γ is the dimensionless overall ionic concentration inside the tissue. The coefficients A's are constants which are related to the cartilage properties and external solution environment [5]. This new formulation is valid for any axisymmetric setup, such as confined or unconfined compressions, perfusion, and indentation tests. The finite difference or finite element solutions for this set of PDEs are straightforward. In present study, finite difference programs for both creep and stress-relaxation indentation tests were developed by using a commercial temporal integrator (Matlab, Mathworks Inc., Natik, MA).

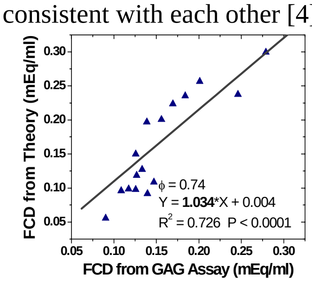
<u>Equilibrium response under dynamic loadings:</u> If the mechanical loading protocol on cartilage tissue is sinusoidal dynamic signal and only the response at equilibrium of the system is desired, the linear governing equations (Eq. (3-6)) can be transferred and further solved in frequency domain without time integration. The four unknown variables can be transformed by assuming:

$$y(t) = y_s \sin\omega t + y_c \cos\omega t \text{, where: } y = \phi, \psi, e, \gamma \quad (7)$$

ω is the dynamic loading frequency. The phase lag of each variable can be obtained from the ratio between y_s and y_c. Numerically, only one matrix inversion is involved to obtain the response of cartilage at each frequency.

RESULTS AND CONCLUSION

Due to the length limit of this abstract, only two results are shown below to illustrate the capability of the above developments. According to correspondence principle, with the known of both intrinsic and apparent mechanical properties, the FCD c_0^F can be easily calculated from Eq. (1). Figure 1 shows the comparison of the calculated FCDs by analyzing the indentation data using correspondence principle and those from biochemical GAG assay. The results from two different methods are remarkably consistent with each other [4].



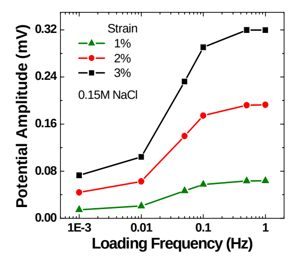


Figure 1 (Left): FCD determination with Eq. (1). **Figure 2** (Right): Electric potential amplitude at cartilage surface under dynamic loadings.

Figure 2 shows amplitude of the electric potential signal on the articular surface increases with increasing loading frequencies. The general trends agree with the observations made in previous experimental studies [5]. In summary, based on the theoretical simplifications at three different levels, a full series of triphasic solutions can be developed which are capable of the predictions of MEC responses in the tissue for various configurations and loading profiles.

REFERENCES: 1. Lai et al., J Biomech Eng, 3: 245-58, 1991. 2. Wan et al., L.Q., et al., Mech & Chem of Biosys, 1: p. 81-99, 2004. 3. Mak et al., J Biomech, 7: 703-14, 1987. 4. Lu et al., J Biomech, BM 3288, 2006. 5. Frank et al., J Biomech, 6: 629-39, 1987.